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Comment on the Equivalence between Fracton/Spectral Dimensionality, and the Dimensionality of Recurrence[1]

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The "fracton/spectral dimensionality" D_{fracton} is an important characteristic of fractal processes. Until now, it has not been interpreted as the fractal dimensionality of any well-defined fractal set, and it has been claimed that D_{fracton} is more "intrinsic" than "the" fractal dimensionality. In fact, D_{fracton} is best understood as an original *dynamical* reinterpretation of a well-defined previously known *kinetic* dimensionality: D_{fracton} is twice the fractal codimensionality of the time instants when a fractal process returns to a point it had previously visited.

The present Comment can be read by itself, but also amplifies of an important point made briefly in my Edinburgh paper.[1] My 1982 book[2] will be referred to as FGN.

Physicists' attention has lately been very much attracted by the important notion of fracton dimensionality D_{fracton}, which was introduced by Alexander and Orbach[3] to describe the *dynamic* properties of random walk on a fractal. This D_{fracton} is the exponent of a spectral distribution, hence is often called *spectral dimensionality.*[4] Given the fractal dimensionality of a cluster, D, and the fractal dimensionality of a random walk on this cluster, D_w, one has $D_{\text{fracton}} = 2D/D_w$. Thus, D_{fracton} is linked to several fractal dimensionalities. But is it the fractal dimensionality of a fully specified set? Thus far, such an interpretation is lacking. In addressing this Conference,[3] Gérard Toulouse presented and discussed a table that called D_{fracton} "intrinsic," while "the" fractal dimensionality D was called "extrinsic." These are fighting words, but in fact *the distrinction drawn by G. Toulouse*

[1] Presented at the *Third Conference on Fractals: Fractals in the Physical Sciences*, held at the National Bureau of Standards, Gaithersburg, Maryland, on November 20–23, 1983.

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[3] While this talk has been widely discussed, its text was not available for inclusion in these proceedings.

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has no substance to fight about. Indeed, the quantity $D_{\rm rec} = 1 - D_{\rm fracton}/2 = 1 - D/D_w$, which is obtained in many calculations[5] as yet another exponent, is a second, temporal, fractal dimensionality, distinct from the fractal's spatial fractal dimensionality.

In other words, G. Toulouse expressed a personal preference between two distinct fractal dimensionalities, that is, between two distinct fractal sets. To elaborate, recall a few facts.[1,2]

A. Each fractal set has a *unique* fractal dimensionality. However, most fractal constructions of interest in physics involve in essential fashion *a number* of distinct fractal sets. (As a matter of fact, we are presently witnessing an explosive growth in the number of distinct fractal sets implicit in each physical problem, and a corresponding growth in the number of different dimensionalities. The resulting clutter is unappealing, but in cannot be eliminated by disregarding some lines in the present long list, only by discovering laws that reduce this list to a shorter one.)

B. Any fractal *process* necessarily combines statics and kinetics, even before dynamics is added. Statics involves one or several fractal sets in real space. As to kinetics, its study by probabilists has long ago learned to focus on a fractal set that belongs to the time axis, namely, the set of the instants when a stochastic process returns to some earlier position (taken as origin). This ancient and vital part of probability theory had given rise to the term "local time" ("time spent in a locus") which my 1977 book[2] relabels less mysteriously as "fractal time." Fractal time had proven essential in numerous investigations; the first were probably those of Elliott W. Montroll, H. Sher, and M. P. Shlesinger.[4] The fractal dimensionality of recurrence, $D_{\rm rec}$, was due to come widely to the fore elsewhere in physics. Unfortunately Toulouse's table failed to include it.

A more complete table immediately raises the question of whether or not the dynamics of a fractal phenomenon is ruled by a dimensionality conceptually distinct from the dimensionality which rules the kinetics. What is most remarkable about the answer[3–5] is that kinetics and dynamics are found to determine each other. Indeed, some well-known results[6] imply in fractal language that the dimensionality of the instants where a random walk on a percolation cluster returns to its point of departure is $D_{\rm rec} = 1 - D_{\rm fracton}/2$. This means that—up to this point in time—*dynamics requires no quantity that was not already involved in the kinetics*. If one so wishes, one can determine everything by combining the fractal dimensionalities D, D_w, and $D_{\rm rec}$, without need for $D_{\rm fracton}$. For the mathematician, it does matter that fraction dimensionality is not yet a fully defined dimensionality

[4] These proceedings include several examples of investigations along these lines, due to J. Bendler, A. Blumen, J. Klafter, G. Zumofen, and M. F. Schlesinger.

of anything, so he will view D_{fracton} as simply the double of familiar notion of codimensionality of the recurrences. In this light, the Alexander and Orbach observation expresses that the dimensionality of recurrence is 1/3. This is a very precise and very challenging mathematical conjecture. On the other hand, the physicist may continue to view the time dimensionality as equal to $1 - D_{\text{fracton}}/2$.

In any event, the claim that D_{fraction} is an entirely new development, and for some purposes is more important than any fractal dimensionality, is unwarranted.

In the context of recurrence, the critical significance of a fracton dimensionality equal to 2 becomes very clear. It corresponds to the classical distinction between random processes that are, respectively, recurrent ($D_{\text{rec}} > 0$, hence $D_{\text{fracton}} < 2$) and transient ($D_{\text{rec}} < 0$, hence $D_{\text{fracton}} > 2$).

Remark on fractal graphs (Section 3.4 of the Edinburgh paper). A fractal graph, often called a hierarchical lattice,[8] is a fractal set in which the distance between two nodes is the "graph distance," defined as the number of links in the shortest paths between the nodes, taken along the static fractal graph itself. This distance has been "reinvented" by many authors, and called "topological" (this term was immediately taken back!), "chemical," "burning," "spreading," "flowing," and the like. Several alternative embeddings in an Euclidean space are often possible; each defines a different fractal set and leads to a different embedding distance as the new flies, hence to a different embedding dimensionality. The dimensionality based on any Euclidean distance is less intrinsic that the graph fractal dimensionality ("chemical, burning, spreading, flowing"). However, there is no indeterminacy whatsoever about the instants of recurrence: they always fall within the time axis, which is the Euclidean straight line, with a single definition of distance. The dimensionality of recurrence is characteristic of the kinetics, and also of the dynamics.

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